

On Collapse of Wave Maps

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Abstract

We derive the universal collapse law of degree 1 equivariant wave maps (solutions of the sigma-model) from the 2+1 Minkowski space-time, to the 2-sphere. To this end we introduce a nonlinear transformation from original variables to blowup ones. Our formal derivations are confirmed by numerical simulations.

1 Introduction

In this paper we investigate the phenomenon of collapse of degree 1 equivariant wave maps (solutions of the σ -model) from the $2 + 1$ Minkowski space-time, \mathbb{R}^{2+1} , to the 2-sphere, S^2 . Besides of purely mathematical interest, the study of the blowup phenomena for such maps is motivated by the recent efforts to understand the singularity formation in general relativity [30].

A wave map, φ , from a $(d+1)$ -dimensional space-time, M , with a metric η to a Riemannian manifold, N , with a metric g , is a critical point of the

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action functional, which, in local coordinates, has the form

$$S(\varphi) := \frac{1}{2} \int g_{AB} \partial_a \varphi^A \partial_b \varphi^B \eta^{ab} \sqrt{-\eta} d^{d+1}x.$$

(This action functional is also known as the σ - model). Critical points of $S(\varphi)$ satisfy the Euler-lagrange equation

$$\partial_a \partial^a \varphi^A + \Gamma_{BC}^A(\varphi) \partial_a \varphi^B \partial^a \varphi^C = 0,$$

where $\Gamma_{BC}^A(\varphi)$ is the Christoffel symbols on N . This system of nonlinear PDEs is Hamiltonian, and in particular has conserved energy, $\mathcal{E}(\varphi)$, and scale invariant in the sense that if $\varphi(x)$ is a solution then so is $\varphi(\lambda x)$. The energy, $\mathcal{E}(\varphi)$, is transformed under scaling as

$$\mathcal{E}(\varphi_\lambda) = \lambda^{2-d} \mathcal{E}(\varphi),$$

where $\varphi_\lambda(x) = \varphi(\lambda x)$. Thus the case $d = 2$ of interest for us is the energy critical.

For a map, φ , to have finite energy, it should converge to a constant at infinity. In this case for each moment of time, t , φ can be extended to a continuous map from S^d to N taking the point at infinity to the limit of $\varphi(x)$ at the spatial infinity. Then one can define the degree, $\deg \varphi$, as the homotopy class of φ as a map from S^d to N . This degree is conserved under the dynamics generated by the Euler-Lagrange equations above.

In our case, M is the $2 + 1$ Minkowski space-time, \mathbb{R}^{2+1} and N is the 2-sphere, S^2 with the standard metric $g := du^2 + \sin^2 u d\theta^2$. In this case the degree of φ is an integer (the degree for maps from S^2 to S^2). Moreover, one has the Bogomolnyi inequality ([7])

$$\mathcal{E}(\varphi) \geq 8\pi |\deg \varphi|,$$

which leads to self-dual and anti-dual equations for the minimizers of the static energy for fixed degrees. These equations have explicit solutions (harmonic or anti-harmonic maps). These solutions will be written out below.

Among the maps of the degree k the simplest, most symmetric maps are the 'radially symmetric' or equivariant maps which are of the form $\varphi_k(\rho, \phi, t) = (u_k(\rho, t), k\phi)$, where (ρ, ϕ) are the polar coordinates in \mathbb{R}^2 and (φ, θ) are the spherical coordinates in S^2 . Then the Euler-Lagrange equation for φ reduces to the equation:

$$\ddot{u} = \Delta u - \frac{k^2}{2\rho^2} \sin(2u) \tag{1}$$

for u_k . Here Δ is the $2D$ spherical Laplacian. Moreover, $\deg \varphi_k = Q(u_k) = k$, where

$$Q(u) := \frac{1}{\pi}(u(\infty) - u(0)).$$

Numerical studies of Eqn (1) led to a conjecture that large-energy, degree one initial data develop singularities in finite time and the singularity formation has the universal form of adiabatic shrinking of the degree-one harmonic map from \mathbb{R}^2 to S^2 [4]. Later, it was shown by Struwe [37] that the existence of a nontrivial harmonic map is in fact the necessary condition for blowup for $2+1$ equivariant wave maps. In this paper we address the question of the dynamics of the blow-up process. We show that there is $0 < t_* < \infty$ such that, as $t \rightarrow t_*$, we have on bounded domains in \mathbb{R}^2

$$u(\rho, t) \approx U(\rho/\lambda(t)),$$

where $U(\rho)$ is the profile of the degree 1 equivariant, static (and in particular harmonic) map, minimizing static energy (see below), and the scaling parameter $\lambda(t)$, satisfies the following second order ODE:

$$\lambda \ddot{\lambda} = \frac{\dot{\lambda}^2}{\ln(\frac{a}{\lambda \dot{\lambda}})}, \text{ with } a = (1.04)^2 e^{-2} \approx .146. \quad (2)$$

We expect that, proceeding as in [5], we can show that the error term in the above relation is $O(\dot{\lambda}^2)$.

Note that Eqn (2) shows that if $\dot{\lambda}|_{t=0} < 0$, then $\dot{\lambda} < 0$, $\ddot{\lambda} > 0$ for $t > 0$ and therefore λ and $|\dot{\lambda}|$ decrease as $t \rightarrow t_*$. Since $\dot{\lambda}^2$ is the small parameter in our analysis (adiabatic regime), our approximation improves as $t \rightarrow t_*$.

An approximate solution of Eqn (2) with two free parameters (constants of integration), t^* and c , is (see Section 6 below)

$$\sqrt{a}(t^* - t) = \lambda e^{\ln^{1/2}(\frac{c}{\lambda})} + c \frac{\sqrt{\pi}}{2} e^{1/4} \left[1 - \Phi(-1/2 + \ln^{1/2}(\frac{c}{\lambda})) \right], \quad (3)$$

where $\Phi(x) \equiv \text{erf}(x)$ is the Fresnel integral [15]. An exact solution of Eqn (2) is obtained in Section 6 (see Eqn (71)). A comparison of the leading term of this solution with a numerical solution of Eqn (1) is given in Fig 1. This figure shows that the two resulting curves are indistinguishable for times sufficiently close to the blow-up time.

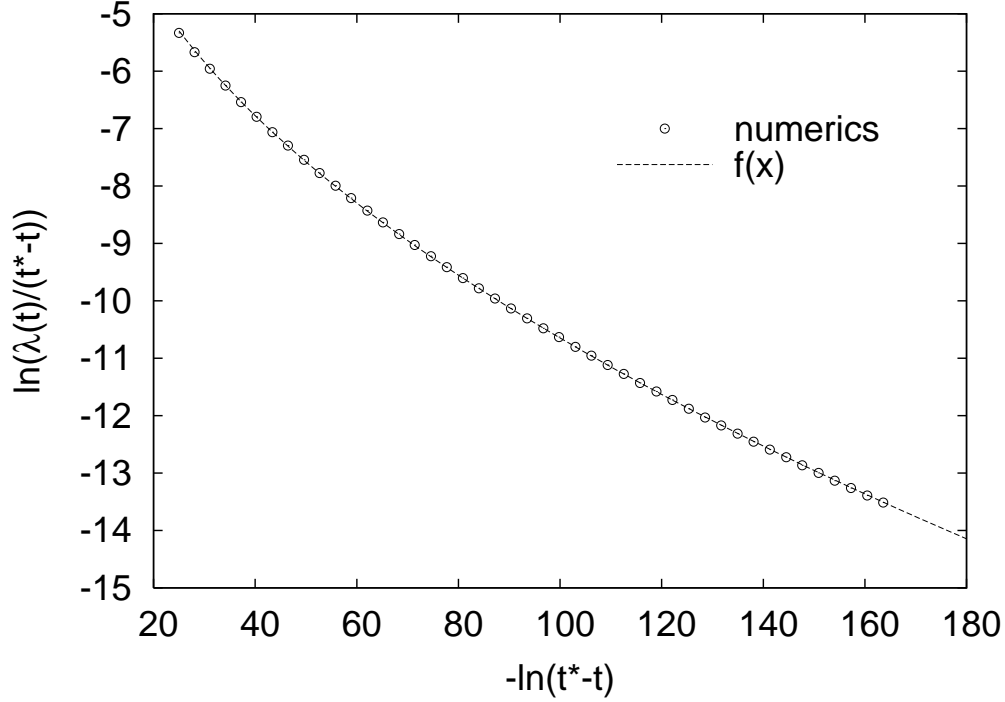


Figure 1: For a numerical solution that blows up at time t^* we plot $y = \ln \frac{\lambda(t)}{t^* - t}$ as a function of $x = -\ln(t^* - t)$ (circles) and compare it with the analytic formula $y = f(x) = \frac{1}{2} \ln(a) - \sqrt{x + b}$, where $a = 0.146$ and b is a free (non-universal) parameter. Fitting b we get an excellent agreement between numerical and analytical results.

Observe that like Eqn (1), Eqn (2) is a Hamiltonian equation. Its Lagrangian is

$$L := h(\dot{\lambda}) - \ln \lambda, \quad (4)$$

where the function h is defined by $h''(x) = -1/f^{-1}(x^2/a)$ with $f(x) = x \ln(1/x)$ (see Section 7).

The local well-posedness for the wave map equations in Sobolev spaces was proven in [17, 18, 19], while the global well-posedness for small initial conditions, in [20, 36, 42, 43, 40, 41] (see also [10, 11, 12, 21, 22, 33, 34, 36, 44, 45]). The research on the problem of blowup for the wave maps started with numerical work [4, 25, 28]. (We do not review here related works for nonlinear wave equations.)

The first numerical evidence for singularity formation for 2+1 equivariant wave maps to the 2-sphere was given in [4]. In this paper (concerned only with $k = 1$ homotopy) the authors showed that blowup has the form of adiabatic shrinking of the harmonic map and formulated conjectures about blowup for large energy, blowup profile and energy concentration and that $\lambda(t)/(T - t)$ must go to zero. As was already mentioned, it was shown rigorously in [37] that the existence of a stationary solution is a necessary condition for the blowup to take place. The blowup scenarios were further numerically investigated in [16, 26] (see references therein for additional works).

The first rigorous results on the blowup rate and profile were obtained in [31, 24]. In particular, [31] has obtained the lower bound on the contraction rate $\lambda(t)$ for $k \geq 4$ wave maps. As it turned out this lower bound conforms exactly to the dynamical law derived for the 4 + 1 Yang-Mills $k = 1$ equivariant solutions in [5] using a formal but careful analysis, explained below in this introduction, justified by numerical computations. (Earlier numerical analysis for the latter model was announced in [6] and described more completely in the survey [2].) (It was noticed in [31] (see below), that the $k \geq 2$ wave map equation are similar to the 4 + 1 Yang-Mills one for $k = 1$.) Finally, for each $b > 1/2$, [24] has constructed special solutions of the $k = 1$ equivariant wave map equation, Eqn (1) with $k = 1$, which blow up at the rate $\lambda(t) \sim (T - t)^{1+b}$.

Equation (1) belongs to a general class of semilinear wave equations in \mathbb{R}^{2+1} of the form

$$\ddot{u} = \Delta u - \frac{1}{\rho^2} f(u). \quad (5)$$

In the case of $f(u) = \frac{k^2}{2} \sin(2u)$, Eqn (5) is, as was already mentioned, the equation for the profile of the equivariant wave map from the 2 + 1 Minkowski space-time of degree k , \mathbb{R}^{2+1} , to the 2-sphere, S^2 . More generally, (5) is satisfied by equivariant maps for the case when N is the surface of revolution with the metric $g := du^2 + g^2(u)d\theta^2$, where $g(u)$ is related to $f(u)$ as $f(u) = g(u)g'(u)$.

In the case of $f(u) = 2u(u^2 - 1)$ the corresponding equation,

$$\ddot{u} = \Delta u - \frac{2}{\rho^2} (u^2 - 1)u, \quad (6)$$

is related to the equation for equivariant Yang-Mills fields of degree 1 in the 4 + 1 dimensions.

Note that

(i) Eqn (5) is invariant with respect to the scaling transformation,

$$u(\rho, t) \rightarrow u\left(\frac{\rho}{\lambda}, \frac{t}{\lambda}\right);$$

(ii) Eqn (5) can be presented as a Hamiltonian system with the standard symplectic form and the Hamiltonian

$$H(u, v) := \int_0^\infty \left(\frac{1}{2}v^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{\rho^2}F(u) \right) \rho d\rho, \quad (7)$$

with $F'(u) = f(u)$. The scaling properties of the Hamiltonian $H(u, v)$ imply that the dimension $d = 2$ is the critical dimension for Eqn (5). This is the dimension treated in this paper.

We assume now that $f(u)$ is a derivative of a *double-well potential* $F(u)$, i.e. $F(u)$ is nonnegative and has at least two global minima, say at a and b for some $b > a$, with $F(a) = F(b) = 0$, and no minima between a and b ($F(u) = \frac{1}{2}\sin^2 u$ and $a = 0, b = \pi$ in the case of $f(u) = \frac{k^2}{2}\sin(2u)$ and $F(u) = \frac{1}{2}(u^2 - 1)^2$ and $a = -1, b = 1$ in the case of $f(u) = 2u(u^2 - 1)$). In this case Eqn (5) has the following features:

- (A) For each $k \in \mathbb{N}$, Eqn (5) has static solutions, $U_k(\rho)$ and $U_{-k}(\rho) = -U_k(\rho)$; they have topological degrees $Q(U_k) = k$ and $Q(U_{-k}) = -k$;
- (B) For $k = 1$, the solution $U_1(\rho)$ is monotonically increasing from a to b , while $U_{-1}(\rho)$ is monotonically decreasing from a to b ;
- (C) The solution, $U_k(\rho)$, is a minimizer of the static energy functional $E(u)$ under the constrain, $Q(u) = k$, on the topological charge;
- (D) Eqn (5) conserves the topological charge $Q(u) := \frac{1}{b-a}(u(\infty) - u(0))$.

Existence of the solutions $U_k(\rho)$ follows from the Bogomolnyi argument, see above. The solutions $U_1(\rho)$ and $U_{-1}(\rho)$ are called the kink solution and anti-kink solution, or simply *kink* and *anti-kink*, respectively. Since the analysis for $k < 0$ can be obtained from analysis for the case $k > 0$ by simply flipping the signs, in what follows we assume that $k > 0$. Note that though Eqn (5) is scale invariant, its static kink solution $U_k(\rho)$ are not. Hence Eqn (5) has an entire family, $U_k(\frac{\rho}{\lambda})$, of kink solutions (symmetry breaking).

From now on *we concentrate on the kink solution*, $U_1(\rho)$ and omit the subindex 1: $U_1(\rho) \equiv U(\rho)$.

There is a feature of Eqn (5) which is not apparent at the first sight but which plays an important role in our analysis of the collapse. The fact that the kink, $U(\rho)$, breaks scale invariance manifests itself in appearance of the dilation zero mode

$$\zeta(\rho) := \frac{1}{2}\rho\partial_\rho U(\rho).$$

This is a zero eigenfunction, $L_\rho\zeta = 0$, for the linearization operator

$$L_\rho = -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} f'(U(\rho)) \quad (8)$$

(negative Fréchet derivative) for the r.h.s. of (5) around $U(\rho)$. This zero mode presents an obstruction to solving Eqn (2) perturbatively, starting with $U(\rho)$, which can be resolved by a modulation theory, provided ζ is an L^2 function, i.e. one can use a Hilbert space spectral theory.

Thus equations of the form (5) can be organized in two classes according to which of the following two properties takes place

- (i) ζ is in L^2
- (ii) ζ is not in L^2 .

The Yang-Mills equation, (6), belongs to the first class while the wave map equation, (1), belong to the second. Indeed, the kink solution for (6) is $U(\rho) = \frac{1-\rho^2}{1+\rho^2}$ and the corresponding zero mode is $\zeta(\rho) = \frac{4\rho^2}{(1+\rho^2)^2}$ (see [2, 5]). For Eqn (1) with $k = 1$ the kink solution is

$$U(\rho) = 2 \arctan \rho,$$

while the scaling zero mode is

$$\zeta(\rho) := \frac{1}{2}\rho\partial_\rho U(\rho) = \frac{\rho}{1+\rho^2}.$$

Clearly, ζ is L^2 in the former case and is not L^2 in the latter case. (This is possible due to the fact that the operator L has no spectral gap: $\sigma(L) = \sigma_{cont}(L) = [0, \infty)$ (see below). The fact that there is a problem with the modulation approach due to the nonintegrability of the zero mode was pointed out by P.Bizoń in 2001, [3].)

Note that $U_k(\rho) = 2 \arctan(\rho^k)$ and the corresponding zero mode is square-integrable for $k > 1$. Thus in this case we expect that at least the formal analysis of [5] of the Yang-Mills equation should go through. (Higher degree equivariant, static solutions for the Yang-Mills equations in $4 + 1$ dimensions, known as instantons, can be found in [1, 32].)

We are interested in solution with initial conditions near the kink manifold

$$M_{kink} := \{U(\rho/\lambda) | \lambda > 0\}.$$

The fact that the square integrability of the zero mode ζ plays an important role in analysis of such solutions can be gleaned from the observation that the effective action $S(U_\lambda)$ on the family $U_\lambda(\rho) := U(\rho/\lambda)$ (the 'effective action' of λ) is equal to

$$S(U_\lambda) = \frac{1}{2} \int \{(\lambda \dot{\lambda})^2 \|\zeta\|^2 - V(U)\} dt,$$

where $V(u) := \frac{1}{2} |\nabla u|^2 + \frac{1}{\rho^2} F(u)$, with $F'(u) = f(u)$, and diverges, if $\zeta \notin L^2$. (For a connection to the geodesic hypothesis see [27, 46]). Here and in what follows $\dot{\lambda} = \frac{\partial \lambda}{\partial t}$.

We present heuristic arguments motivating our approach. It is natural to guess that for an initial condition close to the manifold M_{kink} the solution evolves along this manifold. Let $U(\frac{\rho}{\lambda(t)})$ be the projection of the solution on this manifold. If for this projection $\lambda(t) \rightarrow 0$ as $t \rightarrow t_*$ for some t_* , then the solution collapses at the time t_* . With this in mind we look for solutions to Eqn (5) in the form

$$u(\rho, t) = U(x) + w(x, t), \tag{9}$$

where $x = \rho/\lambda$, a blow-up variable, with λ a slowly varying function of time t (we do not pass to the blow-up time variable). Note that while in a standard approach the scaling, λ , is fixed at the very beginning (with corrections at certain scales possibly considered later on) we leave it free and we look for a differential equation for λ which guarantees that $|w| \ll 1$. However, this simple procedure which works in the case of the Yang-Mills equation mentioned above (see [5, 31]) does not work in the present case as we explain below.

Note that if $\zeta \in L^2$, then $\lambda(t)$ is uniquely determined by the orthogonality condition

$$\langle \zeta, w \rangle = 0. \tag{10}$$

If $\zeta \notin L^2$, then this condition is not well defined unless we assume w belongs to a space of sufficiently fast decaying functions.

Substituting decomposition (9) into (5) leads to the equation for the function w and parameter λ :

$$L_x w + F(w) = -\lambda^2 \frac{\partial^2 U}{\partial t^2}, \quad (11)$$

where $F(w)$ absorbs higher order terms ($F(w) = \lambda^2 \frac{\partial^2 w}{\partial t^2} + N(w)$, $N(w) =$ nonlinearity in w) and L_x is the linearization operator for the r.h.s. of (1) around $U(x)$ given by (8). The operator L_x is self-adjoint. The scaling zero mode, ζ , is a zero mode of this operator: $L_x \zeta = 0$. Since ζ is positive and not L^2 we conclude by the Perron-Frobenius theory that $\sigma(L_x) = [0, \infty)$ and 0 is not an eigenvalue of L_x .

We compute explicitly

$$\begin{aligned} \lambda^2 \frac{\partial^2 U}{\partial t^2} &= \lambda^2 [-2\partial_t(\dot{\lambda}\lambda^{-1})\zeta + 2(\dot{\lambda}\lambda^{-1})^2 x \partial_x \zeta] \\ &= -2\ddot{\lambda}\lambda\zeta + 2\dot{\lambda}^2(\zeta + x\partial_x \zeta). \end{aligned} \quad (12)$$

We multiply Eqn (11) scalarly by $\zeta(x)$. Though ζ is not L^2 one can show using a limiting procedure that $\langle \zeta, L_x w \rangle = 0$, provided $w = o(x)$ and $\partial_x w = o(1)$ at ∞ . Thus we obtain

$$\lambda^2 \langle \zeta, \frac{\partial^2 U}{\partial t^2} \rangle + \langle \zeta, F(w) \rangle = 0. \quad (13)$$

Following [5] we try to develop a perturbation theory in the small parameter $\dot{\lambda}^2$ assuming that term $\lambda\ddot{\lambda}$ is of the order $o(\dot{\lambda}^2)$ (and $\dot{\lambda} < 0$) and similarly for higher order time derivatives of λ , e.g. $\partial_t(\lambda\ddot{\lambda}) = O(\dot{\lambda}^3)$, etc. Furthermore, if our assumption that $|w| \ll 1$ is correct and the integral in $\langle \zeta, F(w) \rangle$ is convergent, then we can drop the term $\langle \zeta, F(w) \rangle$ in (13). Hence we obtain in the leading order $O(\dot{\lambda}^2)$

$$\dot{\lambda}^2 \langle \zeta, \zeta + x\partial_x \zeta \rangle = 0. \quad (14)$$

Considering the integral on the l.h.s. over a bounded domain and integrating by parts one shows that the inner product on the l.h.s. is

$$1/2 \lim_{x \rightarrow \infty} (x^2 \zeta^2(x)). \quad (15)$$

For Eqn (6) this is 0 so we can solve Eqn (11) in the leading order, $w = -\dot{\lambda}^2 L^{-1}(\zeta + x\partial_x \zeta)$. Plugging this result into Eqn (13) and keeping only the terms up to the order $O(\dot{\lambda}^4)$, we obtain the equation for scaling dynamics,

$$\lambda \ddot{\lambda} = \frac{3}{4} \dot{\lambda}^4, \quad (16)$$

in the leading order $O(\dot{\lambda}^4)$ (see [5, 31]). Next, in order to obtain a correction to this equation, we use (13) at the order $O(\dot{\lambda}^6)$ to solve Eqn (11) to the order $O(\dot{\lambda}^4)$ and plug the result to (13). However, at this step we run into logarithmically divergent terms. To overcome this problem we use a multiscale expansion, by introducing an additional scale at infinity (see [5]).

For Eqn (1) with $k = 1$ we have $\lim_{x \rightarrow \infty} (x^2 \zeta^2(x)) = 1$ and so we go to the next term, $-2\ddot{\lambda} \lambda \|\zeta\|^2$, and discover that it diverges logarithmically. Thus for Eqn (1) with $k = 1$ one runs into a problem right away. This shows that decomposition (9) is incompatible with the condition $|w| \ll 1$.

The problem for Eqn (1) with $k = 1$ mentioned above can be also seen in a different but related way. Let us try to solve Eqn (11) by perturbation theory. In the leading order we drop the term $F(w)$ to obtain the leading order approximation to the solution: $w = L^{-1}\varphi$, where $\varphi = -\lambda^2 \frac{\partial^2 U}{\partial t^2}$ and L^{-1} is understood as the Green function of the equation $Lw = \varphi$ (see Section 3). It is easy to check, using Eqns (33) - (34) of Section 2 below, that if $\ddot{\lambda} \neq 0$ then the function $L^{-1}\varphi$ grows at infinity as $x \ln x$, and a straightforward perturbation theory fails. (Not only the correction w is large at ∞ , its energy is infinite.)

The point here is that the function $U(\rho/\lambda)$ is not a good adiabatic solution to Eqn (1) with $k = 1$:

$$\begin{aligned} & \lambda^2 (\partial_t^2 U(x) - \Delta_\rho U(x) + \frac{1}{2\rho^2} \sin(2U(x))) \\ &= -2\ddot{\lambda} \lambda \zeta(x) + \frac{4\dot{\lambda}^2 x}{(1+x^2)^2}, \end{aligned} \quad (17)$$

where $x := \rho/\lambda$ and where we used (12) and the relation $\zeta + x\partial_x \zeta = 2x(1+x^2)^{-2}$. The r.h.s. is not L^2 . The problematic term is $2\ddot{\lambda} \lambda \zeta(x)$. In particular, it leads to the logarithmically divergent term, $2\ddot{\lambda} \lambda \|\zeta\|^2$ in the orthogonality condition. Hence, one has to find a better leading term.

We deal with the problem above by introducing instead of the linear, one-parameter transformation, $\rho \rightarrow \rho/\lambda$, a nonlinear, three-parameter transformation, $\rho \rightarrow f(\rho, \lambda, \alpha, \beta)$, chosen so that $U(f(\rho, \lambda, \alpha, \beta))$ becomes a better

approximate solution to Eqn (1) with $k = 1$ than $U(\rho/\lambda)$. In particular, the problematic term $2\ddot{\lambda}\lambda\zeta$ entering the r.h.s. of Eqn (17) is canceled and therefore the large ρ divergence in Eqn (13) mentioned above is eliminated. Thus, instead of (9), we look for solutions of Eqn (1) in the form

$$u(\rho, t) = U(y) + w(y, t). \quad (18)$$

We consider initial conditions close to $U(y) \equiv U(f(\rho, \lambda, \alpha, \beta))$ (we do not specify the norm, the latter must be determined by a rigorous analysis, see e.g. [31]). After this we proceed as above with Eqn (9). The conditions $|w| \ll 1$ and $w \rightarrow 0$ at $\rho \rightarrow \infty$ and constraints on the energy (7) and its fluctuations imply the differential equation (2) on the parameter $\lambda = \lambda(t)$. We expect that proceeding as in [5] one can obtain corrections to Eqn (2).

The paper is organized as follows. In Section 2 we introduce a change of variables, $\rho \rightarrow f(\rho, \lambda, \alpha, \beta)$, depending nonlinearly on the original variable ρ and on the scaling parameter λ^{-1} (and depending on additional parameters α, β). This is our main new idea. In Section 3 we derive, modulo some technical details which are provided in Appendices 2 and 3, an approximate solution to Eqn (1) with $k = 1$. In Sections 4 and 5 we use an orthogonality condition of the type of (10), the smallness condition on energy fluctuations and the minimum condition on the energy of the approximate solution in order to find our main equation on the scaling parameter λ , Eqn (2). In Section 6 we find exact and approximate solutions of Eqn (2) and in Section 7 we show that this equation is Hamiltonian. In Appendices 1-5 we provide technical calculations used in the main text and explanations of the numerical approaches.

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2 Nonlinear blow-up variables

In this section we introduce a nonlinear, three-parameter (scaling) transformation, $\rho \rightarrow f(\rho, \lambda, \alpha, \beta)$ of the independent spatial variable ρ . This replaces the standard, linear, one-parameter transformation, $\rho \rightarrow \rho/\lambda$. We denote $x := \rho/\lambda$ and define $y = y(x, \lambda\ddot{\lambda}, \alpha, \beta)$ as the solution of the equation

$$\begin{aligned} y &= x - \frac{\lambda\ddot{\lambda}}{2}x^3 \ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y^\alpha x^{1-\alpha}\right), & \text{if } x \leq x_{cr} \\ y &= 2y_{cr} - x + \frac{\lambda\ddot{\lambda}}{2}x^3 \ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y^\alpha x^{1-\alpha}\right), & \text{if } x > x_{cr}, \end{aligned} \quad (19)$$

where $0 \leq \alpha \leq 1$, $\beta > 0$, $x_{cr} = x_{cr}(\lambda, \alpha, \beta)$ and $y_{cr} = y_{cr}(\lambda, \alpha, \beta)$ solve the equations $\frac{\partial}{\partial x}(\text{r.h.s. (19)}) = 0$ and the first equation in (19). We can write x_{cr} and y_{cr} as

$$x_{cr} = \left(\frac{2}{\lambda\ddot{\lambda}}\right)^{1/2} (3 \ln \gamma + 1 - \alpha)^{-1/2} \quad (20)$$

and

$$y_{cr} = \left(\frac{2}{\lambda\ddot{\lambda}}\right)^{1/2} \frac{2 \ln \gamma + 1 - \alpha}{(3 \ln \gamma + 1 - \alpha)^{3/2}}, \quad (21)$$

where $\gamma = \gamma(\alpha, \beta) = \frac{\sqrt{\lambda\ddot{\lambda}}}{\beta}y^\alpha x^{1-\alpha}$ is a solution of the equation

$$\frac{\gamma}{\sqrt{2}} \frac{(3 \ln \gamma + 1 - \alpha)^{1/2+\alpha}}{(2 \ln \gamma + 1 - \alpha)^\alpha} = \beta^{-1}. \quad (22)$$

Eqn (22) is well defined for $\ln \gamma > -\frac{1}{3}(1-\alpha)$ and in this range it has a unique solution for each α and β . We denote $\rho_{cr} := \lambda x_{cr}$.

For $x \leq x_{cr}$ the r.h.s. of (19) is decreasing from ∞ to $-\infty$. Hence the equation (19) has a unique solution for $x \leq x_{cr}$. Furthermore, for $x \geq x_{cr}$ the r.h.s. of (19) increases logarithmically in y from $-\infty$ to ∞ and increases monotonically in x . Since for $x = x_{cr}$ (19) has a unique solution, it has exactly two solutions for $x > x_{cr}$. Of these two solutions we choose the greater one.

Finally, we notice that the function $y = y(x, \lambda\ddot{\lambda}, \alpha, \beta)$ increases monotonically in x for $x > 0$. Indeed, for $0 \leq \alpha \leq 1$, the equations $\frac{\partial}{\partial x}(\text{r.h.s. (19)}) = 0$ and (19) have a unique solution ($x = x_{cr}$, $y = y_{cr}$), and therefore so is the

equation $\frac{\partial y}{\partial x} = 0$. Since y obviously increases monotonically in x for x sufficiently small and sufficiently large, it does so for all $x > 0$.

Write $v(y, t) = u(\rho, t)$, where $y = y(x, \lambda\ddot{\lambda}, \alpha, \beta)$ is given in Eqn (19). In the new variables, Eqn (1) with $k = 1$ becomes

$$-\frac{\partial^2 v}{\partial y^2} - \frac{1}{y} \frac{\partial v}{\partial y} + \frac{\sin(2v)}{2y^2} = \Psi(v), \quad (23)$$

where

$$\Psi(v) := \frac{x^2}{y^2} \left\{ \left(\frac{2y}{x} \chi + \chi^2 \right) \frac{\partial^2 v}{\partial y^2} + \left(\frac{2\chi}{x} + \frac{\partial \chi}{\partial x} \right) \frac{\partial v}{\partial y} - \lambda^2 \frac{\partial^2 v}{\partial t^2} \right\} \quad (24)$$

and $\partial^2/\partial t^2$ is the total derivative in t (i.e. taking into account that y depends on t). Here the function χ is defined according to the equation

$$\frac{\partial y}{\partial x} = \frac{y}{x} + \chi. \quad (25)$$

Eqn (23) is our transformed equation.

Initial conditions for (23) are chosen to be close, in an appropriate norm, to $U(y) \equiv U(y(x, \lambda\ddot{\lambda}, \alpha, \beta))$, where, recall, $U(\rho)$ is the static - kink - solution to Eqn (1). To simplify the exposition we take the initial condition to be just $U(y(x, \lambda\ddot{\lambda}, \alpha, \beta))$.

3 Approximate solution of Eqn (23)

Let $y = y(x, \lambda\ddot{\lambda}, \alpha, \beta)$ be the transformation defined in the previous section. We look for solutions of Eqn (23) in the form

$$v(y, t) = U(y) + w(y, t), \quad (26)$$

where w is a small correction. We plug this decomposition into Eqn (23) to obtain

$$L_y w + N(w) = \Psi(U + w), \quad (27)$$

where operator L_y is defined in Eqn (8), $N(w)$ is the nonlinear in w term defined by this equation and the function $\Psi(v)$ is defined in (24). To find an approximate solution of the latter equation we drop the nonlinearity, $N(w)$, and the term w in $\Psi(U + w)$ to obtain the leading order equation

$$L_y W = \psi, \quad (28)$$

where $\psi(y, t) := \psi(U(y))$. The latter function is given explicitly by

$$\psi(y, t) := \frac{x^2}{y^2} \left\{ \frac{8\chi}{x(1+y^2)^2} + \frac{2}{1+y^2} \left(\frac{\partial\chi}{\partial x} - \frac{2\chi}{x} \right) - \frac{4y\chi^2}{(1+y^2)^2} + \frac{4\lambda^2 y}{(1+y^2)^2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{2\lambda^2}{1+y^2} \frac{\partial^2 y}{\partial t^2} \right\}, \quad (29)$$

where the variable x is connected to y through (19). The counter-term which removes the undesirable term in $\frac{\partial^2 U}{\partial t^2}$ is $\partial\chi/\partial x - 2\chi/x$. As the result we have

$$\psi(y, t) = O\left(\dot{\lambda}^2 \frac{\ln(\lambda \ddot{\lambda} y^2)}{\lambda \ddot{\lambda} y^2}\right) \text{ for } y \geq y_{cr} \quad (30)$$

(see Appendix 3).

Remark 1. *To justify dropping $\frac{\partial^2 w}{\partial t^2}$ from Eqn (27) (it enters into $\psi(U+w)$) is not a simple matter. In a rigorous approach one looks for solutions of (27) in the form*

$$v(y, t) = U(y) + W(y) + \eta(y, t)$$

and shows that the fluctuation η is small (cf. [31]).

Recall that the operator L_y entering Eqn (28) has a zero mode:

$$L_y \zeta = 0. \quad (31)$$

Multiplying Eqn (28) scalarly by ζ and using the self-adjointness of the operator L (and some elementary limiting procedure) and (31), we obtain

$$\int_0^\infty dy y \zeta \psi = 0. \quad (32)$$

This is a (necessary) solvability condition for Eqn (28). It gives an equation on the parameters λ , α and β . (Note that it is an approximate solvability condition for the exact equation (27)).

So far we obtained one equation for the three parameters λ , α and β . To derive another equation we analyze the approximate solution to (27) we obtained: $U(y) + W(y)$, where $W = L^{-1}\psi$ with ψ satisfying (32). Our goal in the rest of this section is to isolate the leading contribution to W . This will be used in the next section to derive the second equation for the parameters.

To find $L^{-1}\psi$ we compute the Green's function for the operator L . Two linearly independent solutions of the homogeneous equation $Lw = 0$ are

$$w_1(y) = \frac{y}{1+y^2} \quad \text{and} \quad w_2(y) = \frac{y}{2} - \frac{1}{2y} + \frac{2y \ln y}{1+y^2} \quad (33)$$

(the first of these solutions is just the scaling zero mode, ζ , the second solution is found in Appendix 1). Hence by the ODE theory

$$L^{-1}\psi = cw_1 + w_1 \int_0^y w_2 \psi y' dy' - w_2 \int_0^y w_1 \psi y' dy' \quad (34)$$

where c is chosen to guarantee solvability of the equation to the second order correction term or by minimizing the energy.

We find the leading contribution to the solution $w = L^{-1}\psi$ of Eqn (28). In what follows we use the following assumptions

$$0 < \lambda \ddot{\lambda} \ll \dot{\lambda}^2 \ll 1, \quad \lambda \partial_t(\lambda \ddot{\lambda}) = O(\dot{\lambda}^3), \quad \lambda \partial_t(\dot{\lambda}^2) = O(\dot{\lambda}^3), \quad \beta = O(1). \quad (35)$$

We see from (20) that

$$x_{cr} = \rho_{cr}/\lambda \sim (\lambda \ddot{\lambda})^{-1/2} \gg 1.$$

Consider first the region $y \leq y_{cr}$. In this region ψ is given by (7), Appendix 3. The latter equation shows that for $y \ll y_{cr}$ the leading part of ψ is

$$\psi_1(y, t) := -\frac{x^3}{y^2} \frac{8\lambda \ddot{\lambda} \left(\ln \left(\frac{\sqrt{\lambda \ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha} \right) + 1/2 \right) - 4\dot{\lambda}^2}{(1+y^2)^2 \left(1 + \frac{\alpha \lambda \ddot{\lambda} x^3}{2y} \right)}, \quad (36)$$

where, recall, $x := \rho/\lambda$ is connected to y through (19). Now, let

$$\psi_2 := \psi - \psi_1. \quad (37)$$

According to (34), the general solution, W , of Eqn (28) decreasing at infinity is of the form

$$\begin{aligned} W(y, t) = & c_1 w_1(y) + w_1(y) \int_0^y w_2(s) \psi(s, t) ds \\ & + w_2(y) \int_y^\infty w_1(s) \psi(s, t) ds, \end{aligned}$$

with $w_1 = \zeta$ and w_2 defined in (33). The function w_2 is singular at $y = 0$. Hence W is bounded only if the condition (32) is satisfied.

The function $\psi_1(y, t)$ is localized on the scale $y \sim 1$, decays as y^{-2} in the region $1 \ll y \ll y_{cr}$ and decays as y^{-5} in the region $y \gg y_{cr}$ (though we are considering at the moment only the region $1 \ll y \ll y_{cr}$, the latter fact allows us to extend the integrals to the entire real axis with a small error). The function ψ_2 is localized at the large scale, $y \sim y_{cr} \gg 1$. After some lengthy computations we find for $y \ll y_{cr}$

$$W(y, t) = \frac{y}{2} \int_0^\infty w_1(s) \psi_1(s, t) s ds + O(\dot{\lambda}^2 \lambda \ddot{\lambda} y^3). \quad (38)$$

Using the expression (36), it is easy to show that

$$\int_0^\infty w_1(s) \psi_1(s, t) s ds = O(\dot{\lambda}^2).$$

4 Energy of the approximate solution and the equation on λ

We compute the energy of our approximate solution $u(\rho, t) = U_{\lambda, \alpha, \beta}(\rho) + W(y)$, where $U_{\lambda, \alpha, \beta}(\rho) := U(y)$, with $y = y(x, \lambda \ddot{\lambda}, \alpha, \beta)$ and U defined in the introduction, and where $W = L^{-1}\psi$, the solution to Eqn (28) (see the previous section). Due to (7), the energy functional is

$$E(u) := \int_0^\infty \left(\frac{1}{2} \dot{u}^2 + \frac{1}{2} |\nabla u|^2 + \frac{1}{2\rho^2} \sin^2 u \right) \rho d\rho. \quad (39)$$

Inserting the approximate solution into this expression, we obtain that $E(U_{\lambda, \alpha, \beta} + W) = E(U_{\lambda, \alpha, \beta}) + \delta E_1$ with

$$\delta E_1 := O(\dot{\lambda}^2 y_{cr}^4) \left(\int_0^\infty w_1(s) \psi_1(s, t) s ds \right)^2 + O(\dot{\lambda}^2). \quad (40)$$

Furthermore, we have that

$$E(U_{\lambda, \alpha, \beta}) = E(U) + \delta E_0 \text{ with } \delta E_0 = O(\dot{\lambda}^2 \ln(1/\dot{\lambda}^2)). \quad (41)$$

We require that the energy correction due to the fluctuation, W , be much smaller than the one due to the modulation:

$$|\delta E_1| \ll |\delta E_0|. \quad (42)$$

Since $\int_0^\infty w_1(s)\psi_1(s,t)sd s = O(\dot{\lambda}^2)$ and $y_{cr} = O(\frac{1}{\sqrt{\lambda\ddot{\lambda}}}) \gg O(\frac{1}{|\dot{\lambda}|})$, this implies that the integral in the leading term in the above expression for δE_1 must vanish:

$$\int_0^\infty dy y \zeta \psi_1 = 0. \quad (43)$$

This gives an implicit equation on the parameters λ, α, β .

In the leading order, we can replace y by $x = \rho/\lambda$ (see the first equation in (19)), so that Eqn (43) becomes

$$\int_0^\infty dx x \frac{x}{1+x^2} \left\{ \frac{8\lambda\ddot{\lambda}x}{(1+x^2)^2} \left[\ln \left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta} x \right) + 1/2 \right] + \frac{4\dot{\lambda}^2 x}{(1+x^2)^2} \right\} = 0. \quad (44)$$

Computing the integrals in (44) (see Appendix 2 for detailed computations), we obtain

$$\dot{\lambda}^2 + 2\lambda\ddot{\lambda} \left[\ln \left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta} \right) + 1 \right] = 0. \quad (45)$$

This is our explicit equation for the parameter λ . It depends on the additional parameter β whose value we still have to determine. Since in the leading approximation ($y \rightarrow x$) the first equation on the r.h.s. of (19) is independent of α , then so are the resulting equations (44) and (45). Eqns (45) and (2) coincide, provided

$$a = \beta^2 e^{-2}. \quad (46)$$

Clearly, solutions of Eqn (45) have the property (35) assumed above. Moreover, if $\lambda(0) > 0$, $\dot{\lambda}(0) < 0$, then, by Eqn (45), $\lambda(t) > 0$, $\dot{\lambda}(t) < 0$, $\ddot{\lambda}(t) > 0$ and therefore $\dot{\lambda}(t)^2 \leq \dot{\lambda}(0)^2$ for $t > 0$. As $t \rightarrow t_*$, $|\dot{\lambda}|$ decreases so that our approximation improves as $t \rightarrow t_*$.

Thus it remains to find the value of the parameter β . To this end we use the condition (32) and minimization of the energy of the leading part of the approximate solution.

5 Values of the parameters α and β

In this section we derive an equation on the parameters α and β and use this equation together with the energy minimization to find the values of these

parameters. We assume (35) and that, at least in the leading approximation,

$$\alpha, \beta \text{ are independent of } t. \quad (47)$$

In what follows we do not display the dependence of the quantities involved on $\lambda\ddot{\lambda}$ (however, the dependence on λ is displayed).

First note that Eqns (32), (37) and (43) imply that

$$\int_0^{y_{cr}} \psi_2 dy + \int_{y_{cr}}^{\infty} \psi dy = 0 \quad (48)$$

From Eqns (21) and (45) we obtain easily

$$\frac{1}{\lambda\ddot{\lambda}} \lambda \frac{\partial(\lambda\ddot{\lambda})}{\partial t} = 2\dot{\lambda} \ln^{-1} \left(\frac{1}{\lambda\ddot{\lambda}} \right) \left[1 + O \left(1/\ln \left(\frac{1}{\lambda\ddot{\lambda}} \right) \right) \right]$$

and

$$\frac{1}{x_{cr}} \lambda \frac{\partial x_{cr}}{\partial t} = -\dot{\lambda} \ln^{-1} \left(\frac{1}{\lambda\ddot{\lambda}} \right) \left[1 + O \left(1/\ln \left(\frac{1}{\lambda\ddot{\lambda}} \right) \right) \right]. \quad (49)$$

As a result in the main approximation in $1/\ln(1/\lambda\ddot{\lambda})$ we should keep in the expressions for the functions ψ and ψ_2 in Eqn (48) only terms proportional to $\dot{\lambda}^2$. The latter terms are given in (7) and (96) in Appendix 3. The most important region in the above integral is where y is of order of y_{cr} ($y_{cr} = O(\frac{1}{\sqrt{\lambda\ddot{\lambda}}}) \gg 1$ due to Eqn (21) and the condition (35)). As a result we can neglect 1 compared to y^2 in (7) and (96). Then Eqn (48) reduces to the equation

$$I(\alpha, \beta) := I_1(\alpha, \beta) + I_2(\alpha, \beta) = 0, \quad (50)$$

where

$$\begin{aligned} I_1 = 2 \int_0^{y_{cr}} \frac{dy x^4}{y^4} \left\{ \frac{\lambda\ddot{\lambda} x^2 \left(\ln \left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha} \right) - \alpha \right)}{y \left(1 + \frac{\alpha\lambda\ddot{\lambda} x^3}{2y} \right)^2} - \frac{2(1-Y^2)}{y \left(1 + \frac{\alpha\lambda\ddot{\lambda} x^3}{2y} \right)^2} + \frac{2(1-Y)}{x \left(1 + \frac{\alpha\lambda\ddot{\lambda} x^3}{2y} \right)} \right. \\ \left. - \frac{\lambda\ddot{\lambda} x}{2} \left(\frac{\alpha x^2}{y^2} \frac{Y^2}{\left(1 + \frac{\alpha\lambda\ddot{\lambda} x^3}{2y} \right)^3} - \frac{6\alpha x}{y \left(1 + \frac{\alpha\lambda\ddot{\lambda} x^3}{2y} \right)^2} - \frac{3(1-\alpha)}{1 + \frac{\alpha\lambda\ddot{\lambda} x^3}{2y}} \right. \right. \\ \left. \left. - \frac{2 \left(1 - \frac{\alpha\lambda\ddot{\lambda} x^3}{y} \right) \left(3 \ln \left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha} \right) + 1 - \alpha \right)}{\left(1 + \frac{\alpha\lambda\ddot{\lambda} x^3}{2y} \right)^2} \right) \right\} \quad (51) \end{aligned}$$

and

$$\begin{aligned}
I_2 = 2 \int_{y_{cr}}^{\infty} dy \frac{x^4}{y^4} & \left\{ \frac{2Y^2}{y \left(1 - \frac{\alpha \lambda \ddot{\lambda} x^3}{2y}\right)^2} + \frac{2Y}{x \left(1 - \frac{\alpha \lambda \ddot{\lambda} x^3}{2y}\right)} \right. \\
& + \frac{\lambda \ddot{\lambda} x}{2} \left[\frac{\alpha x^2}{y^2} \frac{Y^2}{\left(1 - \frac{\alpha \lambda \ddot{\lambda} x^3}{2y}\right)^3} + \frac{6\alpha x}{y \left(1 - \frac{\alpha \lambda \ddot{\lambda} x^3}{2y}\right)^2} - \frac{3(1-\alpha)}{1 - \frac{\alpha \lambda \ddot{\lambda} x^3}{2y}} \right. \\
& \left. \left. - \frac{2 \left(1 + \frac{\alpha \lambda \ddot{\lambda} x^3}{y}\right) \left(3 \ln \left(\frac{\sqrt{\lambda \ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha}\right) + 1 - \alpha\right)}{\left(1 - \frac{\alpha \lambda \ddot{\lambda} x^3}{2y}\right)^2} \right] \right\}. \quad (52)
\end{aligned}$$

Here

$$Y = 1 - \frac{\lambda \ddot{\lambda} x^2}{2} \left(3 \ln \left(\frac{\sqrt{\lambda \ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha} \right) + 1 - \alpha \right). \quad (53)$$

One can further evaluate I_1 and I_2 by changing the variable of integration y in (51) and (5) to z as

$$z = x/x_{cr} = \rho/\rho_{cr}, \quad (54)$$

where x and x_{cr} , as functions of y , $\lambda \ddot{\lambda}$, α , β are given in (19) and the definitions following this equation, and compute the resulting integral numerically. In particular, one can show that for $\alpha = 0$, $I_1(\alpha = 0, \beta) = 1$, $I_2(\alpha = 0, \beta) = 0$ and therefore $I(\alpha = 0, \beta) = 1$, independently of the value of β . Thus we cannot take $\alpha = 0$ in our transformation (19).

We chose the parameters α and β which minimize the energy $E(\alpha, \beta) := E(U_{\lambda, \alpha, \beta})$, where, recall, $U_{\lambda, \alpha, \beta}(\rho) := U(y(x, \lambda \ddot{\lambda}, \alpha, \beta))$, given that the equation (50), $I(\alpha, \beta) = 0$, holds. To find these minimizers we use Eqns (39) and $U(\rho) = 2 \arctan \rho$ to rewrite the energy $E(\alpha, \beta)$ as

$$E(\alpha, \beta) = 2 \int_0^\infty d\rho \rho \frac{1}{(1+y^2)^2} \left\{ \left(\frac{\partial y}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial \rho} \right)^2 + \frac{y^2}{\rho^2} \right\}. \quad (55)$$

We find numerically (see Appendix 5 for the analytical part) that the energy $E(\alpha, \beta)$ is minimized on the curve $I(\alpha, \beta) = 0$ at the point

$$\beta_0 = 1.04 \text{ and } \alpha_0 = 0.65436. \quad (56)$$

This is a special point for the curve $I(\alpha, \beta) = 0$. Our numerics show that while the functions $\alpha = \alpha(\beta)$ and $\beta = \beta(\alpha)$ determined by the equation $I(\alpha, \beta) = 0$ are double-valued, their branches originate exactly at this point (and form a wedge there). So the equation $I(\alpha, \beta) = 0$ has a unique solution only for $\beta = \beta_0$ or for $\alpha = \alpha_0$ and has no solutions for $\beta > \beta_0$ or for $\alpha < \alpha_0$.

Substituting $\beta = \beta_0 = 1.04$ into Eqn (45), we obtain the following value for the parameter a :

$$a = 0.146.$$

This proves Eqn (2) with $a = 0.146$.

6 Investigation of Eqn (2)

In this section we find an approximate solution to Eqn (2) (which is, up to a redefinition of the parameters, the equation (45)). Iterating this equation, we find, in the leading approximation, the following equation

$$\frac{\ddot{\lambda}}{\dot{\lambda}} = \frac{\dot{\lambda}}{\lambda \ln(a/\dot{\lambda}^2)}. \quad (57)$$

Solution of the Eqn (57) with two free parameters of integration, $c > 0$ and t^* , is

$$\sqrt{a} (t^* - t) = \int_0^\lambda dx e^{\ln^{1/2}(\frac{c}{x})}. \quad (58)$$

Changing the variable of integration as $\ln\left(\frac{c}{x}\right) = z^2$, we reduce Eqn (58) for the parameter λ to the form Eqn (3) given in Introduction.

Now we derive an exact expression for a general solution of (2). We introduce the function $f(x) := x \ln(1/x)$. For $0 < x < e^{-1}$ this function has the inverse, $f^{-1}(x)$. Using this inverse we rewrite Eqn (2) as

$$\frac{\lambda \ddot{\lambda}}{a} = f^{-1}\left(\frac{\dot{\lambda}^2}{a}\right). \quad (59)$$

(Note that for $x \rightarrow 0$, $f^{-1}(x) = \frac{x}{\ln x} + \dots$, so in the leading approximation of (59) gives (57).) Integrating equation (59) gives

$$\ln \lambda = F(\dot{\lambda}) \text{ where } F(y) = \frac{1}{2} \int^{\frac{y^2}{a}} \frac{dz}{z} g(z), \quad (60)$$

with $g(z) := z/f^{-1}(z)$. Using the equation $f(f^{-1}(y)) = y$, or, more explicitly, $f^{-1}(y) \ln(1/f^{-1}(y)) = y$, we find that the function $g(z)$ satisfies the equation

$$g(z) = \ln\left(\frac{g(z)}{z}\right). \quad (61)$$

Differentiating the latter equation, we find

$$g'(z) = -\frac{g(z)}{z(g(z) - 1)}. \quad (62)$$

Using this equation we integrate

$$\int^x \frac{dz}{z} g(z) = - \int^x dz g'(z) (g(z) - 1) = -\frac{1}{2} (g(x) - 1)^2 + \text{const.} \quad (63)$$

This gives

$$F(y) = -\frac{1}{4} (g(y^2/a) - 1)^2 + \text{const}, \quad (64)$$

which together with Eqn (60) yields

$$g\left(\frac{\dot{\lambda}^2}{a}\right) = 1 + 2\sqrt{\ln\left(\frac{c}{\dot{\lambda}}\right)} \quad (65)$$

for some constant c . The latter equation can be integrated as follows

$$\sqrt{a}(t^* - t) = \int_0^\lambda \frac{dx}{\left[g^{-1}\left(1 + 2\sqrt{\ln\left(\frac{c}{x}\right)}\right)\right]^{1/2}}. \quad (66)$$

Next we find the function $g^{-1}(x)$. The definition of the function $f(x)$ implies $f(e^{-x}) = xe^{-x}$, which yields

$$\frac{xe^{-x}}{f^{-1}(xe^{-x})} = x, \quad (67)$$

which, in turn, leads to $g(xe^{-x}) = x$, which finally gives the expression

$$g^{-1}(x) = xe^{-x}. \quad (68)$$

Now Eqns (66) and (68) imply

$$\sqrt{a}(t^* - t) = \int_0^\lambda dx \frac{e^{1/2 + \sqrt{\ln(\frac{c}{x})}}}{\sqrt{1 + 2\sqrt{\ln(\frac{c}{x})}}}. \quad (69)$$

Changing the variable of integration as $\ln(c/x) = z^2$ we find

$$\sqrt{a}(t^* - t) = \sqrt{2}ce^{1/2} \int_{\sqrt{\ln(c/\lambda)}}^\infty dz \frac{z}{\sqrt{z + 1/2}} e^{z - z^2}. \quad (70)$$

We obtain for $\sqrt{\ln(c/\lambda)} \gg 1$ the approximate expression

$$\begin{aligned} \sqrt{a}(t^* - t) = \frac{\lambda}{\sqrt{2}[\ln(c/\lambda)]^{1/4}} e^{1/2 + \sqrt{\ln(c/\lambda)}} \\ \times \left[1 + \frac{1}{4\sqrt{\ln(c/\lambda)}} - \frac{1}{32} \frac{1}{\ln(c/\lambda)} \right]. \end{aligned} \quad (71)$$

Eqn (3) is an approximation for this exact expression, it differs from the latter by a slowly varying factor which can be found in the next approximation to (3).

7 Hamiltonian Formulation

Eqn (2) is a Hamiltonian system. Indeed, it can be obtained from the Lagrangian

$$L = h(\dot{\lambda}) - \ln \lambda \quad (72)$$

where the function h is defined by

$$h''(x) = -\frac{1}{f^{-1}(x^2/a)} \quad (73)$$

with $f(x) = x \ln(1/x)$ (see Section 5). Now the generalized momentum, Hamiltonian and energy can be found in the standard way. In particular, the energy is given by

$$E = -\dot{\lambda} \frac{\partial L}{\partial \dot{\lambda}} + L = -\dot{\lambda} h'(\dot{\lambda}) + h(\dot{\lambda}) - \ln \lambda. \quad (74)$$

This is the energy conservation law. On the other hand, differentiating Eqn (74) w.r.to t , we obtain the equation of motion (2).

7. Conclusion

We presented detailed arguments that for an open set of initial conditions close to the degree 1 equivariant, static wave map, the solutions of the wave map equation (σ -model) collapse in a finite time. Near the collapse point the solutions have a universal profile given by the modified the degree 1 equivariant, static wave map depending on a time-dependent parameter λ . This parameter describes the rate of compression (scaling) of the collapse profile. We derived a second order Hamiltonian dynamical equation for the scaling parameter, λ . We also found approximate solutions of this equation. These solutions are of a rather complex form. They are in an excellent agreement with direct numerical simulations of the wave map equation.

Appendix 1

To solve the equation $Lw = g$, we should find first of all two linear independent solution of linear equation

$$L_x w = 0. \quad (75)$$

The first solution of this equation is the scaling zero mode ζ

$$w_1 = \zeta = \frac{x}{1 + x^2}. \quad (76)$$

The second solution w_2 satisfies the inhomogeneous equation of first order:

$$w_1 w_2' - w_2 w_1' = \frac{1}{x}. \quad (77)$$

The standard solution of this equation is

$$w_2 = w_1 z; \quad z' = x + \frac{2}{x} + \frac{1}{x^3}; \quad z = C + \frac{x^2}{2} + 2 \ln x - \frac{1}{2x^2}. \quad (78)$$

Setting $C = 0$, we obtain

$$w_2 = \frac{x}{2} + \frac{2x \ln x}{1+x^2} - \frac{1}{2x}. \quad (79)$$

To obtain general solution of the equation $L_x w = g$, we rewrite it as a first order ODE

$$\frac{\partial}{\partial x} \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{1}{x^2} \left(1 - \frac{8x^2}{(1+x^2)^2}\right) & -\frac{1}{x} \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} - g \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (80)$$

Two linear independent solutions of A1.5 are

$$\begin{pmatrix} w_1 \\ w'_1 \end{pmatrix}, \quad \begin{pmatrix} w_2 \\ w'_2 \end{pmatrix}. \quad (81)$$

By the method of variation of constants we look for a general solution of inhomogeneous Eqn (80) in the form

$$\begin{pmatrix} w \\ v \end{pmatrix} = c_1 \begin{pmatrix} w_1 \\ w'_1 \end{pmatrix} + c_2 \begin{pmatrix} w_2 \\ w'_2 \end{pmatrix}. \quad (82)$$

where $c_{1,2}$ are functions of x . Inserting (82) into Eqn (80), we find

$$\frac{\partial c_1}{\partial x} = x w_2 g; \quad \frac{\partial c_2}{\partial x} = -x w_1 g. \quad (83)$$

Appendix 2

To derive Eqn (45) from Eqn (44) we should calculate two simple integrals. One of them is

$$\int_0^\infty \frac{dy y^3}{(1+y^2)^3} = \frac{1}{2} \int_0^\infty \frac{dx x}{(1+x)^3} = \frac{1}{4}. \quad (84)$$

The second integral is ($\varepsilon \rightarrow 0$)

$$\begin{aligned}
\int_0^\infty \frac{dy}{(1+y^2)^3} y^3 \ln y &= \frac{1}{4} \int_0^\infty \frac{dx}{(1+x)^3} x \ln x = -\frac{1}{4} \int_\varepsilon^\infty \ln x \, d\left(\frac{1}{x+1} - \frac{1}{2} \frac{1}{(x+1)^2}\right) \quad (85) \\
&= \frac{1}{8} \ln \varepsilon + \frac{1}{4} \int_\varepsilon^\infty \frac{dx}{x} \left(\frac{1}{x+1} - \frac{1}{2} \frac{1}{(x+1)^2}\right) \\
&= \frac{1}{8} \ln \varepsilon + \frac{1}{8} \int_\varepsilon^\infty dx \left[\frac{1}{x} - \frac{1}{x+1} + \frac{1}{(x+1)^2}\right] = \frac{1}{8}.
\end{aligned}$$

Using the values of these two integrals, we obtain Eqn (45) from Eqn (44).

Appendix 3

Now we will find an explicit expression for the inhomogeneous term ψ . We consider separately two domains $\{y \leq y_{cr}\} \equiv \{\rho \leq \rho_{cr}\}$ and $\{y \geq y_{cr}\} \equiv \{\rho \geq \rho_{cr}\}$. First, we compute $\frac{\partial y}{\partial t}$ and $\frac{\partial^2 y}{\partial t^2}$.

Recall the notation $x := \rho/\lambda$ and

$$Y := 1 - \frac{\lambda \ddot{\lambda} x^2}{2} \left(3 \ln \left(\frac{\sqrt{\lambda \ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha} \right) + 1 - \alpha \right) \quad (86)$$

and let

$$A := \ln \left(\frac{\sqrt{\lambda \ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha} \right), \quad X := 1 + \frac{\alpha \lambda \ddot{\lambda} x^3}{2y} \quad \text{and} \quad Z := 1 - \frac{\alpha \lambda \ddot{\lambda} x^3}{2y} \quad (87)$$

In the domain $y \leq y_{cr}$ we have

$$\begin{aligned}
\lambda \frac{\partial y}{\partial t} &= -\dot{\lambda} x \frac{Y}{X} + O\left(x^3 \lambda \frac{\partial}{\partial t} (\lambda \ddot{\lambda})\right), \\
\lambda^2 \frac{\partial^2 y}{\partial t^2} &= \left(2\dot{\lambda}^2 - \ddot{\lambda} \lambda\right) x \frac{Y}{X} + \frac{\dot{\lambda} x^3}{2} \lambda^3 \frac{\partial}{\partial t} \left[\frac{\lambda \ddot{\lambda}}{\lambda^2} \frac{3A + 1 - \alpha + \frac{\alpha x}{y}}{X} \right] \\
&\quad + O\left(x^3 \lambda \dot{\lambda} \frac{\partial}{\partial t} (\lambda \ddot{\lambda})\right) \quad (88)
\end{aligned}$$

where

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\frac{\lambda \ddot{\lambda}}{\lambda^2} \frac{3A + 1 - \alpha + \frac{\alpha x}{y}}{X} \right] \\ &= \frac{\dot{\lambda} \lambda \ddot{\lambda}}{\lambda^3} \left\{ \frac{\alpha x^2}{y^2} \frac{Y^2}{X^3} - \frac{6\alpha x}{yX^2} - \frac{3(1-\alpha)}{X} - \frac{2(2Z-1)(3A+1-\alpha)}{X^2} \right\}. \end{aligned} \quad (89)$$

In the domain $y \geq y_{cr}$ we have

$$\begin{aligned} \lambda \frac{\partial y}{\partial t} &= \frac{1}{Z} \left\{ \dot{\lambda} x Y + O \left(x^3 \lambda \frac{\partial}{\partial t} (\lambda \ddot{\lambda}) \right) \right\}, \\ \lambda^2 \frac{\partial^2 y}{\partial t^2} &= - \left(2\dot{\lambda}^2 x - \lambda \ddot{\lambda} x \right) \frac{Y}{Z} - \frac{\dot{\lambda} x^3}{2} \lambda^3 \frac{\partial}{\partial t} \left[\frac{\lambda \ddot{\lambda}}{\lambda^2} \frac{3A + 1 - \alpha - \frac{\alpha x}{y}}{Z} \right] \\ &\quad + O \left(x^3 \lambda \dot{\lambda} \frac{\partial}{\partial t} (\lambda \ddot{\lambda}) \right), \end{aligned} \quad (90)$$

where

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\frac{\lambda \ddot{\lambda}}{\lambda^2} \frac{3A + 1 - \alpha - \frac{\alpha x}{y}}{Z} \right] \\ &= \frac{\dot{\lambda} \lambda \ddot{\lambda}}{\lambda^3} \left\{ \frac{\alpha x^2}{y^2} \frac{Y^2}{Z^3} + \frac{6\alpha x}{yZ^2} - \frac{3(1-\alpha)}{Z} - \frac{2(2X-1)(3A+1-\alpha)}{Z^2} \right\}. \end{aligned} \quad (91)$$

Now we present an explicit form of the function χ entering the definition of ψ , (29), and introduced in (25). Due to Eqn (19) we have

$$\chi = \begin{cases} -\lambda \ddot{\lambda} x^2 (A + 1/2) X^{-1}, & x < x_{cr} \\ \left[-\frac{2y_{cr}}{x} + \lambda \ddot{\lambda} x^2 (A + 1/2) \right] Z^{-1}, & x > x_{cr}. \end{cases} \quad (92)$$

Next, we give here an explicit expression for the expression $\partial\chi/\partial x - 2\chi/x$. We compute

$$\frac{\partial\chi}{\partial x} - \frac{2\chi}{x} = -\frac{\lambda \ddot{\lambda} x^2}{X} \left\{ \frac{1-\alpha}{x} + \frac{\alpha Y}{yX} - \frac{\alpha \lambda \ddot{\lambda} x^2}{2yX} (A + 1/2) \left(3 - \frac{x}{y} \frac{Y}{X} \right) \right\}, \quad (93)$$

for $x < x_{cr}$, and

$$\begin{aligned} \frac{\partial \chi}{\partial x} - \frac{2\chi}{x} = \frac{6y_{cr}}{x^2 Z} + \frac{\lambda \ddot{\lambda} x^2}{Z} \left(\frac{1-\alpha}{x} - \frac{\alpha Y}{y Z} \right) \\ + \frac{\alpha \lambda \ddot{\lambda} x}{2y Z^2} \left(-2y_{cr} + \lambda \ddot{\lambda} x^3 (A + 1/2) \right) \left(3 + \frac{x Y}{y Z} \right), \quad (94) \end{aligned}$$

for $x > x_{cr}$.

Note, that the function y , Eqn (19), is chosen so as to cancel the term $-2\lambda \ddot{\lambda} x / (\lambda^2(1+y^2))$ arising from the last term in expression (29) (see the first term on the r.h.s. of (88) and the first term on the r.h.s. of (17)). With the help of Eqns (29), (93) and (88) we obtain following expression for the function ψ in the domain $x \leq x_{cr}$:

$$\begin{aligned} \psi = \frac{x^2}{y^2} \left\{ -\frac{8\lambda \ddot{\lambda} x}{(1+y^2)^2} \frac{A+1/2}{X} - \frac{4\dot{\lambda}^2 x}{(1+y^2)^2 X} + \frac{2\dot{\lambda}^2 \lambda \ddot{\lambda} x^4 y (A-\alpha)}{(1+y^2)^2 X^2} \right. \\ + \frac{2\alpha \lambda \ddot{\lambda} x^2}{y X (1+y^2)} \left[\frac{y}{x} - \frac{Y}{X} + \frac{\lambda \ddot{\lambda} x^2}{2} \frac{A+1/2}{X} \left(3 - \frac{x Y}{y X} \right) \right] \\ - \frac{4(\lambda \ddot{\lambda})^2 x^4 y (A+1/2)^2}{(1+y^2)^2 X^2} - \frac{4\dot{\lambda}^2 y x^2 (1-Y^2)}{(1+y^2)^2 X^2} \\ - \frac{2x}{1+y^2} \left[-\left(2\dot{\lambda}^2 - \lambda \ddot{\lambda} \right) \frac{1-Y}{X} + \frac{\dot{\lambda}^2 \lambda \ddot{\lambda} x^2}{2} \left(\frac{\alpha x^2 Y^2}{y^2 X^3} - \frac{6\alpha x}{y \left(1 + \frac{\alpha \lambda \ddot{\lambda} x^3}{2y} \right)^2} \right. \right. \\ \left. \left. - \frac{3(1-\alpha)}{X} - \frac{2(2Z-1)(3A+1-\alpha)}{X^2} \right) \right] \right\}. \quad (95) \end{aligned}$$

Now, using Eqns (92), (29), (93), (7), we find expression for function ψ in the domain $y \geq y_{cr}$. In fact, to obtain the equation on the parameter λ we need to know only the part of ψ in $\{y \geq y_{cr}\}$, proportional to $\dot{\lambda}^2$. For this reason we write out only this part:

$$\begin{aligned} \psi = 2\dot{\lambda}^2 \frac{x^3}{y^4} \left\{ \frac{2x Y^2}{y Z^2} + \frac{2Y}{Z} + \frac{\lambda \ddot{\lambda} x^2}{2Z} \left(\frac{\alpha x^2 Y^2}{y^2 Z^2} + \frac{6\alpha x}{y Z} - 3(1-\alpha) \right. \right. \\ \left. \left. - \frac{2(2X-1)(3A+1-\alpha)}{Z} \right) \right\} + \text{term proportional to } \lambda \ddot{\lambda}. \quad (96) \end{aligned}$$

Finally, we show Eqn (30) which was stated in Section 3. Indeed, the definitions of Y and Z and the second equation in (19) imply that for $y \geq y_{cr}$

$$\frac{Y}{Z} = \frac{3\lambda\ddot{\lambda}x^2}{2\alpha}B^3, \quad Y \sim \lambda\ddot{\lambda}x^2B, \quad y \sim \lambda\ddot{\lambda}x^3B,$$

where $B := \ln(\lambda\ddot{\lambda}x^2)$. Using these relations and the equation (96), we arrive at the desired relation (30).

Appendix 4

In this appendix we compute the partial derivatives of energy $E = E(\alpha, \beta)$ w.r.to parameters α, β . Using expression (55), we obtain

$$\begin{aligned} \frac{\partial E}{\partial \beta} = 4 \int_0^\infty d\rho \frac{\rho}{(1+y^2)^2} & \left\{ \frac{\partial y}{\partial t} \frac{\partial}{\partial \beta} \left(\frac{\partial y}{\partial t} \right) + \frac{\partial y}{\partial \rho} \frac{\partial}{\partial \beta} \left(\frac{\partial y}{\partial \rho} \right) + \frac{y}{\rho^2} \frac{\partial y}{\partial \beta} \right. \\ & \left. - \frac{2y}{1+y^2} \frac{\partial y}{\partial \beta} \left(\left(\frac{\partial y}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial \rho} \right)^2 + \frac{y^2}{\rho^2} \right) \right\} \quad (97) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial E}{\partial \alpha} = 4 \int_0^\infty d\rho \frac{\rho}{(1+y^2)^2} & \left\{ \frac{\partial y}{\partial t} \frac{\partial}{\partial \alpha} \left(\frac{\partial y}{\partial t} \right) + \frac{\partial y}{\partial \rho} \frac{\partial}{\partial \alpha} \left(\frac{\partial y}{\partial \rho} \right) + \frac{y}{\rho^2} \frac{\partial y}{\partial \alpha} \right. \\ & \left. - \frac{2y}{1+y^2} \left(\left(\frac{\partial y}{\partial t} \right)^2 + \left(\frac{\partial y}{\partial \rho} \right)^2 + \frac{y^2}{\rho^2} \right) \right\}. \quad (98) \end{aligned}$$

Recall the notation

$$X := 1 + \frac{\alpha\lambda\ddot{\lambda}x^3}{2y} \quad \text{and} \quad Z := 1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y} \quad (99)$$

and let

$$X_{cr} := 1 + \frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}} \quad \text{and} \quad Z_{cr} := 1 - \frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}. \quad (100)$$

From Eqn (19) we find

$$\frac{\partial y}{\partial \beta} = \frac{\lambda\ddot{\lambda}x^3}{2\beta} \frac{1}{X}, \quad x < x_{cr} \quad (101)$$

$$\frac{\partial y}{\partial \beta} = \frac{\lambda \ddot{\lambda}}{2\beta} \frac{1}{Z} \left[\frac{2x_{cr}^3}{X_{cr}} - x^3 \right], \quad x > x_{cr}$$

and

$$\frac{\partial y_{cr}}{\partial \beta} = \frac{\lambda \ddot{\lambda} x_{cr}^3}{2\beta} \frac{1}{X_{cr}}.$$

Using Eqn (101), we obtain the time derivative of $\frac{\partial y}{\partial \beta}$ in the leading approximation in $1/\ln\left(\frac{1}{\lambda\lambda}\right)$:

$$\frac{\partial}{\partial t} \left(\frac{\partial y}{\partial \beta} \right) = -\frac{\dot{\lambda} \lambda \ddot{\lambda} x^3}{2\beta \lambda Z^2} \left[3 + \frac{\alpha \lambda \ddot{\lambda} x^4 Y}{2y^2 X} \right], \quad x < x_{cr}, \quad (102)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial y}{\partial \beta} \right) = -\frac{\dot{\lambda} \lambda \ddot{\lambda}}{2\beta \lambda Z} \left\{ \frac{3 + \frac{\alpha \lambda \ddot{\lambda} x^4 Y}{2y^2 Z}}{Z} \left(\frac{2x_{cr}^3}{X_{cr}} - x^3 \right) - \frac{6x_{cr}^3}{X_{cr}} \right\}, \quad x > x_{cr}. \quad (103)$$

In a similar way we find derivative of y w.r. to α :

$$\frac{\partial y}{\partial \alpha} = -\frac{\lambda \ddot{\lambda} x^3 \ln\left(\frac{y}{x}\right)}{2 X}, \quad x < x_{cr}, \quad (104)$$

$$\frac{\partial y}{\partial \alpha} = -\frac{\lambda \ddot{\lambda}}{2} \frac{1}{Z} \left[-x^3 \ln\left(\frac{y}{x}\right) + 2x_{cr}^3 \frac{\ln\left(\frac{y_{cr}}{x_{cr}}\right)}{X_{cr}} \right], \quad x > x_{cr},$$

$$\frac{\partial y_{cr}}{\partial \alpha} = -\frac{\lambda \ddot{\lambda} x_{cr}^3 \ln\left(\frac{y_{cr}}{x_{cr}}\right)}{2 X_{cr}}.$$

Taking the time derivative of Eqn (104), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial \alpha} \right) = & -\frac{\dot{\lambda} \lambda \ddot{\lambda} x^3}{2\lambda X^2} \left\{ \frac{\lambda \ddot{\lambda} x^3}{y} \left(\ln\left(\frac{\sqrt{\lambda \ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha}\right) + 1/2 \right) - \right. \\ & \left. \ln\left(\frac{y}{x}\right) \left(3 + \frac{\alpha \lambda \ddot{\lambda} x^4 Y}{2y^2 X} \right) \right\}, \quad x < x_{cr}, \quad (105) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial \alpha} \right) = & -\frac{\dot{\lambda}\ddot{\lambda}\ddot{\lambda}}{2\lambda Z} \left\{ -\frac{3 + \frac{\alpha\lambda\ddot{\lambda}x^4}{2y}\frac{Y}{Z}}{Z} \left[2x_{cr}^3 \frac{\ln\left(\frac{y_{cr}}{x_{cr}}\right)}{X_{cr}} - x^3 \ln\left(\frac{y}{x}\right) \right] + \frac{6x_{cr}^3 \ln\left(\frac{y_{cr}}{x_{cr}}\right)}{X_{cr}} \right. \\ & \left. - \frac{x^3}{yZ} \left(2y_{cr} - \lambda\ddot{\lambda}x^3 \left(\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha}\right) + 1/2 \right) \right) \right\}, \quad x > x_{cr}. \quad (106) \end{aligned}$$

Note that the main contribution to the partial derivatives (97) and (98) comes from the domain $x \sim x_{cr}$. Both derivatives are sums of terms proportional to $\dot{\lambda}^2$ and to $\lambda\ddot{\lambda}$. The coefficients for these terms are of order of 1. As result, since we assumed that $|\lambda\ddot{\lambda}| \ll \dot{\lambda}^2$, we have to find in the expressions for (97) and (98) only the terms proportional to $\dot{\lambda}^2$.

Using Eqns (101) - (106) we can write the r.h.s. of (97) and (98) in a more explicit form

$$\begin{aligned} \frac{1}{4} \frac{\partial E}{\partial \beta} = & \dot{\lambda}^2 \int_0^{x_{cr}} \frac{dx x^5 \lambda \ddot{\lambda}}{2\beta y^4} \frac{Y}{\left(1 + \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}\right)^3} \left[3 + \frac{\alpha\lambda\ddot{\lambda}x^4}{2y^2} \frac{Y}{1 + \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}} - \frac{2xY}{y} \right] \\ & - \dot{\lambda}^2 \int_{x_{cr}}^{\infty} \frac{dx x^2 \lambda \ddot{\lambda}}{2\beta y^4} \frac{Y}{\left(1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}\right)^2} \left\{ \left[\frac{1}{1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}} \left(3 + \frac{\alpha\lambda\ddot{\lambda}x^4}{2y^2} \frac{Y}{1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}} \right) \right. \right. \\ & \times \left(\frac{2x_{cr}^3}{1 + \frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}} - x^3 \right) - \frac{6x_{cr}^3}{1 + \frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}} \left. \right] + \frac{2x}{y} \frac{Y}{1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}} \left(\frac{2x_{cr}^3}{1 + \frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}} - x^3 \right) \left. \right\}, \quad (107) \end{aligned}$$

$$\begin{aligned} \frac{1}{4} \frac{\partial E}{\partial \alpha} = & \dot{\lambda}^2 \int_0^{x_{cr}} \frac{dx x^5 \lambda \ddot{\lambda} Y}{2y^4 \left(1 + \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}\right)^3} \left\{ \frac{\lambda\ddot{\lambda}x^3}{y} \left(\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha}\right) + 1/2 \right) \right. \\ & \left. - \ln\left(\frac{y}{x}\right) \left(3 + \frac{\alpha\lambda\ddot{\lambda}x^4}{2y^2} \frac{Y}{1 + \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}} \right) + \frac{2x}{y} \ln\left(\frac{y}{x}\right) Y \right\} + \dot{\lambda}^2 \int_{x_{cr}}^{\infty} \frac{dx x^2 \lambda \ddot{\lambda} Y}{2y^4 \left(1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}\right)^2} \\ & \times \left\{ \frac{1}{1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}} \left(3 + \frac{\alpha\lambda\ddot{\lambda}x^4}{2y^2} \frac{Y}{1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}} \right) \left(\frac{2x_{cr}^3 \ln\left(\frac{y_{cr}}{x_{cr}}\right)}{1 + \frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}} - x^3 \ln\left(\frac{y}{x}\right) \right) \right. \\ & \left. - \frac{6x_{cr}^3 \ln\left(\frac{y_{cr}}{x_{cr}}\right)}{1 + \frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}} + \frac{x^3}{y \left(1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}\right)} \left(2y_{cr} - \lambda\ddot{\lambda}x^3 \left(\ln\left(\frac{\sqrt{\lambda\ddot{\lambda}}}{\beta} y^\alpha x^{1-\alpha}\right) + 1/2 \right) \right) \right\} \end{aligned}$$

$$+\frac{2x}{y} \frac{Y}{1 - \frac{\alpha\lambda\ddot{\lambda}x^3}{2y}} \left(\frac{2x_{cr}^3 \ln\left(\frac{y_{cr}}{x_{cr}}\right)}{1 + \frac{\alpha\lambda\ddot{\lambda}x_{cr}^3}{2y_{cr}}} - x^3 \ln\left(\frac{y}{x}\right) \right) \Bigg\}. \quad (108)$$

Let $\alpha = \alpha(\beta)$ be a solution of Eqn (50). We find numerically (see Appendix 5) that β changes on the interval $(0, \beta_0]$, where $\beta_0 = 1.0405$ (the corresponding value of α is $\alpha_0 = 0.65436$.) Using expressions for $\frac{\partial E}{\partial \alpha}$ and $\frac{\partial E}{\partial \beta}$, derived above, we show numerically that the function

$$\Phi := \frac{\partial E}{\partial \beta} + \frac{\partial E}{\partial \alpha} \frac{\partial \alpha}{\partial \beta} \quad (109)$$

is negative for $\beta = \beta_0$ and for $\beta \rightarrow 0$, with $E(\alpha, \beta)$ having absolute minimum at $\beta = \beta_0$.

Appendix 5

Numerical calculations with help of Eqns (51) and (5) show that there is a point (α_0, β_0) ,

$$\beta_0 = 1.0405, \alpha_0 = .6543626,$$

so that the equation (50), $I(\alpha, \beta) = 0$, has no solution for $\alpha < \alpha_0$ and for $\beta > \beta_0$. Moreover, the solution of the equation $I(\alpha, \beta) = 0$ for β determines a double-valued function $\beta = \beta(\alpha)$, whose branches coalesce at $\alpha = \alpha_0$ and have different derivatives there (see Eqns (110) and (111) below). Moreover, $I(\alpha, \beta) = 0$ has the unique solution β_0 at $\alpha = \alpha_0$. Hence the solution of the equation $I(\alpha, \beta) = 0$ for α also leads to a double-valued function $\alpha = \alpha(\beta)$.

Numerical calculations give the following expansions for the lowest branch,

$$\beta = \beta_0 - \beta_1(\alpha - \alpha_0) - \beta_2(\alpha - \alpha_0)^2, \quad (110)$$

$\alpha > \alpha_0$, and for the distance, Δ , between the branches along the α -axis,

$$\Delta = \gamma_1(\beta_0 - \beta) - \gamma_2(\beta_0 - \beta)^2, \quad (111)$$

where

$$\beta_1 = 2.54732, \beta_2 = 13.8297, \quad (112)$$

$$\gamma_1 = .08029, \gamma_2 = .42736. \quad (113)$$

(Solving (110) for α gives the lower branch of the function $\alpha = \alpha(\beta)$. Adding (111) to this solution gives the upper branch of $\alpha = \alpha(\beta)$.)

To find the second "end" point on the α -interval we check the point $\alpha = 1$ where the dependence of y on x in Eqn (19) can be found in an explicit form. To do this we note that (21) and (22) with $\alpha = 1$ imply that

$$\gamma = \frac{\sqrt{\lambda \ddot{\lambda} y_{cr}}}{\beta}, \quad \beta^2 = \frac{8}{27\gamma^2 \ln \gamma} \quad \text{and} \quad \lambda \ddot{\lambda} y_{cr}^2 = \frac{8}{27 \ln \gamma}. \quad (114)$$

We also have $y_{cr}/x_{cr} = 2/3$. For $\alpha = 1$ solvability condition of Eqn (19) is

$$\gamma > e^{1/2}. \quad (115)$$

Indeed, set

$$y = y_{cr} z, \quad z = 1 + \delta, \quad \frac{x}{y_{cr}} = \frac{3}{2} + \tau \quad (116)$$

In the range $0 < \delta \ll 1$ we have

$$\frac{2}{3}\tau^2 = \delta \left(1 - \frac{1}{2 \ln \gamma} \right) \quad (117)$$

From this equation we see that β should satisfy the inequality given in Eqn (115).

Now we set $y = y_{cr} z$. For $z < 1$ we obtain from the first equation in (19), with $\alpha = 1$, and from (114) the following cubic equation for the ratio $\frac{x}{y_{cr}}$

$$\frac{4}{27} \left(\frac{x}{y_{cr}} \right)^3 \frac{\ln(\gamma z)}{\ln \gamma} - \frac{x}{y_{cr}} + z = 0. \quad (118)$$

Solution of Eqn (118) in the range $z < 1$ is

$$\frac{x}{y_{cr}} = 3 \left(\frac{\ln \gamma}{\ln(1/(\gamma z))} \right)^{1/2} \sinh \left[\frac{1}{3} \ln \left(z \sqrt{\frac{\ln(1/(\gamma z))}{\ln \gamma}} + \sqrt{1 + z^2 \frac{\ln(1/(\gamma z))}{\ln \gamma}} \right) \right] \quad (119)$$

for $\gamma z < 1$ and

$$\frac{x}{y_{cr}} = 3 \sqrt{\frac{\ln \gamma}{\ln(\gamma z)}} \sin \left[\frac{1}{3} \arctan \frac{z \sqrt{\ln(\gamma z)/\ln \gamma}}{\sqrt{1 - z^2 \ln(\gamma z)/\ln \gamma}} \right] \quad (120)$$

for $\gamma z > 1$.

In the range $z \geq 1$ the ratio x/y_{cr} solves the following cubic equation (see the second equation in (19))

$$\frac{4}{27} \frac{\ln(\gamma z)}{\ln \gamma} \left(\frac{x}{y_{cr}} \right)^3 - \frac{x}{y_{cr}} + 2 - z = 0. \quad (121)$$

Let z_0 be solution of equation

$$1 = (z_0 - 2)^2 \frac{\ln(\gamma z_0)}{\ln \gamma}. \quad (122)$$

We split the semi-interval $z > 1$ into two sub-intervals. In the interval $1 < z < z_0$ we have

$$\frac{x}{y_{cr}} = 3 \sqrt{\frac{\ln \gamma}{\ln(\gamma z)}} \sin \phi, \quad (123)$$

where

$$\begin{aligned} \phi &= \frac{\pi}{6} + \frac{1}{3} \arctan \left(\frac{\sqrt{1 - (2 - z)^2 \ln(\gamma z) / \ln \gamma}}{(2 - z) \sqrt{\ln(\gamma z) / \ln \gamma}} \right), 1 < z < 2, \\ \phi &= \frac{\pi}{3} + \frac{1}{3} \arctan \left(\frac{(z - 2) \sqrt{\ln(\gamma z) / \ln \gamma}}{\sqrt{1 - (z - 2)^2 \ln(\gamma z) / \ln \gamma}} \right), 2 < z < z_0. \end{aligned} \quad (124)$$

In the range $z > z_0$ we have

$$\frac{x}{y_{cr}} = \frac{3}{2} (Q^{1/3} + \frac{\ln \gamma}{\ln(\gamma z)} Q^{-1/3}) \quad (125)$$

where

$$Q = (z - 2) \frac{\ln \gamma}{\ln(\gamma z)} + \sqrt{\left((z - 2) \frac{\ln \gamma}{\ln(\gamma z)} \right)^2 - \left(\frac{\ln \gamma}{\ln(\gamma z)} \right)^3} \quad (126)$$

Using Eqns (119)-(126) we obtain with help of numerical calculations, that Eqn (50) at $\alpha = 1$ has solution only as β goes to zero. This means that $\alpha = 1$ is the second end point of the α -interval.

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